OSCILLATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this note we study the zeros of solutions of differential equations of the form $u'' + pu = 0$. A criterion for oscillation is found, and some sharper forms of the Sturm comparison theorem are given.

§1. Number of zeros.

Consider the linear differential equation

$$u''(x) + p(x)\,u(x) = 0\,, \qquad \text{where} \quad p(x) = \frac{1}{(1-x^2)^2}\,, \tag{1}$$

on the interval $-1 < x < 1$. Two independent solutions are

$$\sqrt{1-x^2} \qquad \text{and} \qquad \sqrt{1-x^2}\,\log\frac{1+x}{1-x}\,,$$

so it is clear that no solution of the differential equation (1) can vanish more than once in the interval $(-1,1)$, unless it vanishes identically. This property was a key to Nehari's study of sufficient conditions for univalence of an analytic function in the unit disk [6,7,8].

The function $p(x)$ in (1) has a remarkable feature. If the differential equation is perturbed to $u'' + Cpu = 0$ for an arbitrary constant $C > 1$, then every solution has infinitely many zeros in $(-1,1)$. Indeed, if we write $C = 1 + \delta^2$, then a pair of linearly independent solutions is given by

$$\sqrt{1-x^2}\,\cos\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right) \qquad \text{and} \qquad \sqrt{1-x^2}\,\sin\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right)\,, \tag{2}$$

from which our statement follows.

This curious phenomenon, the abrupt change in behavior of solutions as C passes through the value 1, seems to call for closer inspection. By symmetry, it suffices

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to study solutions over the interval $[0,1)$. It is natural to consider a differential equation of the form

$$u''(x) + \frac{1+\sigma(x)}{(1-x^2)^2}\, u(x) = 0\,, \qquad 0 \le x < 1\,, \tag{3}$$

where $\sigma(x)$ is a positive continuous function with $\lim_{x\to 1-} \sigma(x) = 0$, and to ask what asymptotic behavior of the function $\sigma(x)$ gives rise to solutions with finitely or infinitely many zeros in the interval $[0,1)$. The following theorem gives a fairly complete answer.

Theorem 1. *Let $\sigma(x)$ be a positive continuous function on $[0,1)$ with $\sigma(x) \to 0$ as $x \to 1$. Let*

$$\lambda = \liminf_{x\to 1} \left(\log \frac{1}{1-x}\right)^2 \sigma(x)\,, \qquad \Lambda = \limsup_{x\to 1} \left(\log \frac{1}{1-x}\right)^2 \sigma(x)\,.$$

(i) If $\lambda > 1$, then each solution of (3) has infinitely many zeros in $[0,1)$.
(ii) If $\Lambda < 1$, then each nontrivial solution of (3) has at most a finite number of zeros in $[0,1)$.
(iii) If $\lambda \le 1 \le \Lambda$, then the number of zeros may be finite or infinite.

It should be remarked that if one nontrivial solution of a differential equation $u'' + qu = 0$ has infinitely many zeros, then all solutions do. This is an immediate consequence of Sturm's classical theorem on the interlacing of zeros of any pair of independent solutions. (See for instance [1], Chapter 2.)

Before discussing the proof of Theorem 1, we want to give an application. A *Nehari function* is a positive continuous even function $p(x)$ on the interval $(-1,1)$ for which $(1-x^2)^2 p(x)$ is nondecreasing on $[0,1)$ and the differential equation $u'' + pu = 0$ has a nonvanishing solution on $(-1,1)$. Nehari functions arise in connection with Nehari's general univalence criterion [7], expressed in terms of the Schwarzian derivative. Examples are $p(x) = (1-x^2)^{-2}$, $p(x) = 2(1-x^2)^{-1}$, and $p(x) = \pi^2/4$, with respective nonvanishing solutions $u = \sqrt{1-x^2}$. $u = 1-x^2$, and $u = \cos(\pi x/2)$. For any Nehari function, it is clear that the index

$$\mu = \lim_{x\to 1-} (1-x^2)^2 p(x)$$

exists and $\mu \ge 0$. It can be shown [2] that $\mu \le 1$, and in fact that $\mu < 1$ unless $p(x) = (1-x^2)^{-2}$. As a simple application of the Sturm comparison theorem, we showed in [2] that for a constant $C > 0$ the solutions of $u'' + Cpu = 0$ have infinitely many zeros in $(-1,1)$ if $C\mu > 1$ and finitely many if $C\mu < 1$. The case $C\mu = 1$ is indeterminate in general, but we can apply Theorem 1 to classify one special example. For any parameter t in the interval $1 < t < 2$, consider the function

$$p(x) = \frac{t\left(1-(t-1)x^2\right)}{\left(1-x^2\right)^2}\,.$$

It is a Nehari function with nonvanishing solution $u = (1-x^2)^{t/2}$ and index $\mu = t(2-t)$. Take $C = 1/t(2-t)$, so that $C\mu = 1$ and

$$Cp(x) = \frac{1+\sigma(x)}{(1-x^2)^2}\,, \qquad \text{where} \;\; \sigma(x) = \frac{(t-1)(1-x^2)}{2-t}\,.$$

Then the function $\sigma(x)$ satisfies the requirements of Theorem 1, and $\lambda = \Lambda = 0$, so solutions of the equation $u'' + Cpu = 0$ can have only a finite number of zeros in $(-1,1)$.

The proof of Theorem 1 is based on the following lemma.

Relative Convexity Lemma. *Let p and q be continuous functions on an interval $[a,b)$, where $b \le \infty$. Let u and v be solutions of the respective differential equations $u''+pu = 0$ and $v''+qv = 0$. Suppose that $u(x) > 0$ in $[a,b)$ and define the function*

$$F(x) = \int_a^x \frac{1}{u(t)^2}\,dt\,, \qquad a \le x < b\,. \tag{4}$$

Then F is continuous and increasing on $[a,b)$, and it maps this interval onto an interval $[0,\ell)$, where $0 < \ell \le \infty$. Let G denote the inverse of F. Then the function

$$w(y) = \frac{v(G(y))}{u(G(y))} \tag{5}$$

satisfies the differential equation

$$w''(y) = [p(x) - q(x)]\,u(x)^4\,w(y)\,, \qquad x = G(y)\,, \;\; 0 \le y < \ell\,. \tag{6}$$

In particular, if $v(x) > 0$ and $p(x) \le q(x)$ on $[a,b)$, then $w''(y) \le 0$, so that the function w is concave on the interval $[0,\ell)$.

The lemma is proved by straightforward differentiation. Details may be found for instance in [2].

Proof of Theorem 1. It is easy to check that $u(x) = \sqrt{1-x^2}$ is a solution of the differential equation (1) that does not vanish in $[0,1)$. The corresponding function (4) is found to be $F(x) = L(x)$, where

$$L(x) = \int_0^x \frac{1}{1-t^2}\,dt = \tfrac{1}{2}\log\frac{1+x}{1-x}\,, \qquad 0 \le x < 1\,.$$

If $v(x)$ is a solution of the equation (3), then the equation (6) for w reduces to

$$w''(y) + h(y)w(y) = 0\,, \qquad \text{where} \;\; h(y) = \sigma(G(y))\,, \;\; 0 \le y < \infty\,. \tag{7}$$

We now use the Sturm comparison theorem (see [1], Chapter 2) to study the zeros of w, which are the same as those of v after precomposition with G. We compare (7) with the differential equation

$$W''(y) + \frac{c}{y^2} W(y)\,, \qquad c > 0\,, \tag{8}$$

whose solutions, as we will see, begin to exhibit infinitely many zeros in $[1, \infty)$ when $c > \frac{1}{4}$. Indeed, the function $W(y) = y^\alpha$ solves the equation (8) provided $\alpha(\alpha - 1) + c = 0$, so that linearly independent sets of solutions are

$$\begin{aligned} &\{y^{\alpha_1}\,, y^{\alpha_2}\}\,, \qquad \alpha_1\,, \alpha_2 = \tfrac{1}{2}(1 \pm \sqrt{1-4c})\,, \qquad \text{if } \ c < \tfrac{1}{4}\,; \\ &\{\sqrt{y}\,, \sqrt{y}\log y\}\,, \qquad \text{if } \ c = \tfrac{1}{4}\,; \\ &\{\sqrt{y}\cos(\beta \log y)\,, \sqrt{y}\sin(\beta \log y)\}\,, \qquad \beta = \tfrac{1}{2}\sqrt{4c-1}\,, \qquad \text{if } \ c > \tfrac{1}{4}\,. \end{aligned} \tag{9}$$

Infinitely many zeros occur only when $c > \frac{1}{4}$. By Sturm comparison, it then follows that the solutions of (7) will have finitely many zeros in $[1, \infty)$ if

$$\limsup_{y \to \infty} \ y^2 h(y) < \tfrac{1}{4}$$

and infinitely many if

$$\liminf_{y \to \infty} \ y^2 h(y) > \tfrac{1}{4}\,.$$

Since $h(y) = \sigma(G(y))$ and $y = L(x)$, parts (i) and (ii) of the theorem follow. In the next section we give some examples in support of (iii). $\square$

In fact, the comparison theorem shows that only finitely many zeros occur if $L(x)^2\sigma(x) \le \frac{1}{4}$ for all $x < 1$ sufficiently near the point 1.

Hartman [3] states a variant of Theorem 1 as an exercise (Chapter XI, Exercise 1.2). Hille ([4], p. 461) describes related results. In any event, we believe our approach via the relative convexity lemma clarifies the issue and simplifies the proof.

The theorem can be refined by further applications of the relative convexity lemma. If $L(x)^2\sigma(x) \to \frac{1}{4}$, the solutions of (3) can be classified as oscillatory or nonoscillatory according to the rate of approach. To be more precise, suppose that $y^2 h(y) = \frac{1}{4} + \tau(y)$, where $\tau(y) \to 0$ as $y \to \infty$. Observe that $\omega(y) = \sqrt{y}$ is a nonvanishing solution of $y^2\omega''(y) + \frac{1}{4}\omega(y) = 0$, and the corresponding integral (4) is $f(y) = \int_1^y 1/s\,ds = \log y$, with inverse $y = g(t) = e^t$. Let $w(y)$ be any solution of $y^2 w''(y) + (\frac{1}{4} + \tau(y))w(y) = 0$, and define $z(t) = w(e^t)/\omega(e^t)$. By the relative convexity lemma, $z''(t) + \tau(e^t)z(t) = 0$. As in the proof of the theorem, we see that $z(t)$ has finitely many zeros in $(0, \infty)$ if $\limsup_{t\to\infty} t^2\tau(e^t) < \frac{1}{4}$ and infinitely many if $\liminf_{t\to\infty} t^2\tau(e^t) > \frac{1}{4}$. Accordingly, this classifies the solutions of (3) with $L(x)^2\sigma(x) = \frac{1}{4} + \tau(L(x))$. Hille [4] and Hartman [3] describe similar refinements.

We turn now to a second application of the relative convexity lemma. By way of motivation, recall that p and q are continuous and $p(x) < q(x)$ on $[a, b)$, and if u and v are respective solutions of $u'' + pu = 0$ and $v'' + qv = 0$ with the same initial conditions at a, then by the Sturm comparison theorem, the first zero of v must occur before that of u. More precisely, if $u(x_1) = 0$ at some point $x_1 \in (a, b)$, then $v(x_2) = 0$ at some $x_2 \in (a, x_1)$. However, if $u(x) > 0$ on $[a, b)$ and $u(x) \to 0$ as $x \to b$, the function v need not vanish anywhere in (a, b). For example, consider on the interval $[0, 1)$ the one-parameter family of functions

$$p_t(x) = \frac{t}{(1-x^2)^2}\,, \qquad 0 < t < 1\,.$$

The solution u_t to $u'' + p_t u = 0$ with initial conditions $u_t(0) = 1\,,\ u_t'(0) = 0$ is

$$u_t(x) = \tfrac{1}{2}\sqrt{1-x^2}\left\{\left(\frac{1+x}{1-x}\right)^{\gamma} + \left(\frac{1-x}{1+x}\right)^{\gamma}\right\}, \qquad \gamma = \tfrac{1}{2}\sqrt{1-t} \tag{10}$$

(*cf.* Kamke [5], page 492, formula 2·369). If $0 < t < s < 1$, then $p_t(x) < p_s(x)$ and $0 < u_s(x) < u_t(x)$ for $0 < x < 1$, yet $u_s(1) = u_t(1) = 0$. The following theorem tells us precisely when this kind of behavior can occur.

Theorem 2. *Let p be a continuous function on an interval $[a, b)$, where $b \le \infty$. Let u be a solution of the differential equation $u'' + pu = 0$ such that $u(x) > 0$ on $[a, b)$. In terms of u, define the function F as in (4). Let q be a continuous function with $q(x) \ge p(x)$ but $q(x) \not\equiv p(x)$ on $[a, b)$, and let v be the solution of $v'' + qv = 0$ with the same initial data $v(a) = u(a)$ and $v'(a) = u'(a)$. Then in order that v vanish at some point in (a, b) for every such choice of function q, it is necessary and sufficient that $F(x) \to \infty$ as $x \to b$.*

Proof. Suppose first that $F(x) \to \infty$ as $x \to b$. Then F is an increasing function that maps the interval $[a, b)$ onto $[0, \infty)$. Let $G = F^{-1}$ and consider the function $w(y)$ as defined in (5). Simple calculations show that $w(0) = 1$ and $w'(0) = 0$. But if $v(x) > 0$ on (a, b), then w is a nonconstant concave function on $[0, \infty)$, by the relative convexity lemma and the hypothesis that $q(x) \ge p(x)$ but $q(x) \not\equiv p(x)$. It follows that w must vanish somewhere on $(0, \infty)$, because $w(0) > 0$ and $w'(0) = 0$. Hence v vanishes somewhere on (a, b).

Conversely, suppose that $F(x) \to \ell < \infty$ as $x \to b$. Then we will show that there are permissible choices of the function $q(x) \ge p(x)$ for which the corresponding solution v does not vanish on (a, b). Let $q(x) = p(x) + r(x)$, where $r(x) \ge 0$ but $r(x) \not\equiv 0$. Equation (6) then takes the form

$$w''(y) + r(x)\,u(x)^4\,w(y) = 0\,, \qquad x = G(y)\,, \quad 0 \le y < \ell\,.$$

If we choose r so that $r(x)u(x)^4 \le \pi^2/4\ell^2$, then by the Sturm comparison theorem, the function w cannot vanish in the interval $(0, \ell)$, since the solution to $W'' +$

$(\pi^2/4\ell^2)W = 0$ with the given initial data is simply $W(y) = \cos(\pi y/2\ell)$, which has no zeros in $(0, \ell)$. Hence the corresponding solution v has no zeros in (a, b). $\square$

The relative convexity lemma can also be applied to *derive* the solutions (2) and (10) of the equation $u'' + Cpu = 0$, where $p(x) = (1 - x^2)^{-2}$ and $C > 1$ or $0 < C < 1$, given only the nonvanishing solution $u(x) = \sqrt{1 - x^2}$ in the case $C = 1$. Then the function (4) is again $F(x) = L(x)$, and for any solution of $v'' + Cpv = 0$ the differential equation (6) reduces to $w'' = (1-C)w$. For $C > 1$ the last equation has solutions $w(y) = \sin(\delta y)$ and $\cos(\delta y)$, where $\delta = \sqrt{C - 1}$. For $0 < C < 1$ it has solutions $w(y) = \exp\left(\pm\sqrt{1 - C}\, y\right)$. With the substitution $y = F(x)$, this leads to the expressions in (2) and (10).

§2. Examples.

Some examples will now be offered in support of the assertions in part (*iii*) of Theorem 1. We show first that the condition $\lambda < 1$ does not prevent an infinite number of zeros, even when $\Lambda = 1$.

We begin by constructing a function σ with $\lambda = \Lambda = 1$ for which the solutions of (3) have infinitely many zeros. To define σ it suffices to construct the function $h(y)$ that occurred in the proof of Theorem 1, since $\sigma(x) = h(F(x))$. We do this in such a way that on large disjoint intervals $I_n \subset [0, \infty)$, the function h has the form

$$h(y) = \frac{c_n}{y^2}\,, \qquad c_n = \frac{1}{4} + \frac{1}{n^2}\,.$$

The corresponding quantity β in (9) is equal to $1/n$, so that if $I_n = [a_n, b_n]$ with $b_n/a_n > e^{n\pi}$, then by Sturm comparison any solution of (8) will have a zero in I_n. We construct the intervals $I_1, I_2, \ldots$ inductively so that $a_{n+1} > b_n > e^{n\pi} a_n$. On the intervening intervals (b_n, a_{n+1}) we extend the definition of the function $h(y)$ by linear interpolation. Then $\lim_{y\to\infty} y^2 h(y) = \frac{1}{4}$, so that $\lambda = \Lambda = 1$ and the solutions of (3) have infinitely many zeros.

The preceding construction can be modified to give λ any value in the interval $(0, 1)$. For any prescribed number α with $0 < \alpha < \frac{1}{4}$, we can define $h(y)$ as a continuous function in such a way that in each of the intervals (b_n, a_{n+1}) the quantity $y^2 h(y)$ dips below the value $\frac{1}{4}$ with a minimum equal to α. Then

$$\liminf_{y\to\infty} y^2 h(y) = \alpha \quad \text{and} \qquad \limsup_{y\to\infty} y^2 h(y) = \tfrac{1}{4}\,.$$

Thus $\lambda = 4\alpha < 1$ and $\Lambda = 1$, whereas the solutions of (3) have infinitely many zeros.

Next we construct an example with $\lambda = 1$ and $\Lambda > 1$ arbitrarily prescribed, for which some solution of the differential equation (3) vanishes only once in $(0, 1)$. Consequently, no solution can have more than 2 zeros. For any β in the interval $\frac{1}{4} < \beta \leq \infty$, it will be enough to construct a continuous function $h(y)$ with $\liminf_{y\to\infty} y^2 h(y) = \frac{1}{4}$ and $\limsup_{y\to\infty} y^2 h(y) = \beta$ such that some solution of the equation (7) has only one zero in $(0,\infty)$. Let $I_n = (a_n, b_n)$ be disjoint intervals in $(1,\infty)$, with $b_n < a_{n+1}$ and $a_n \to \infty$. Choose a sequence of numbers $\beta_n > \frac{1}{4}$ with $\beta_n \to \beta$ as $n \to \infty$. On each interval I_n, let $\frac{1}{4} \le y^2 h(y) \le \beta_n$ with $y^2 h(y) = \beta_n$ at the midpoint, and set $y^2 h(y) = \frac{1}{4}$ elsewhere in $(0,\infty)$. Outside the intervals I_n the solutions to $w'' + hw = 0$ have the form $w(y) = \sqrt{y}(A + B\log y)$, with different values of the constants A and B in each component. Let $w = \sqrt{y}(1 + \log y)$ in $(0, a_1]$ and write $w = \sqrt{y}(A_n + B_n \log y)$ for $y \in [b_n, a_{n+1})$. For each n it is clear that the differences $|A_{n+1} - A_n|$ and $|B_{n+1} - B_n|$ can be made arbitrarily small provided the length $b_n - a_n$ is sufficiently small. Once a sequence $\{a_n\}$ is chosen, we can select the points b_n inductively so that $A_n, B_n > \frac{1}{2}$ and so that w remains positive on I_n. Then w remains positive on $[1,\infty]$, and on the interval $[0,1]$ it will vanish exactly once. Because w is concave when $w(y) > 0$, this will guarantee that the solution remains positive on $[1,\infty)$. On the interval $(0,1)$ it will vanish exactly once.

§3. An integral criterion.

Consider now the differential equation $u''(x) + p(x)u(x) = 0$ on the real line $-\infty < x < \infty$, where $p(x)$ is an even continuous function. What properties of p will ensure that every solution has a zero? The problem reduces to consideration of the special solution with $u(0) = 1$ and $u'(0) = 0$. Indeed, this is an even function, so if it vanishes once it will vanish twice, and then every other solution will vanish somewhere in between, by Sturm's theorem on the interlacing of zeros.

Hence it is enough to let $p(x)$ be continuous on $0 \le x < \infty$ and to ask whether the solution with initial data $u(0) = 1$ and $u'(0) = 0$ vanishes somewhere on $(0,\infty)$. This will certainly be true if $p(x) > 0$. Then $u''(0) < 0$ and so $u'(x) < 0$ in some interval $(0,\delta]$, and the solution is concave so long as $u(x) > 0$, so it must lie below its tangent line constructed at the point $(\delta, u(\delta))$. This tangent line has negative slope and so it cuts across the x-axis. Consequently, the solution $u(x)$ must do the same.

The following theorem says that the condition $p(x) > 0$ can be relaxed to require only that the function have a positive integral. We adopt the notation

$$p(x)^+ = \max\{p(x), 0\}\,, \qquad p(x)^- = \max\{-p(x), 0\}\,,$$

so that $p(x) = p(x)^+ - p(x)^-$.

Theorem 3. *Suppose that $p(x)$ is continuous on the interval $[0,\infty)$, and let $u(x)$ be the solution of the differential equation $u'' + pu = 0$ with initial conditions $u(0) = 1$ and $u'(0) = 0$. If $\int_0^\infty p(x)^- dx < \int_0^\infty p(x)^+ dx \le \infty$, then $u(x) = 0$ at some point $x \in (0,\infty)$.*

Proof. Suppose, on the contrary, that $u(x) > 0$ throughout the interval $(0,\infty)$, and consider its logarithmic derivative $\varphi = u'/u$. Then $\varphi(0) = 0$ and

$$\varphi'(x) = \frac{u''(x)}{u(x)} - \left(\frac{u'(x)}{u(x)}\right)^2 = -p(x) - \varphi(x)^2 \le -p(x)\,.$$

Thus by hypothesis,

$$\varphi(x) = \int_0^x \varphi'(t)\,dt \le -\int_0^x p(t)\,dt = \int_0^x \left(p(t)^+ - p(t)^-\right)\,dt < -\varepsilon$$

for some $\varepsilon > 0$ if $x \ge b_0$, a sufficiently large positive number. For $x \ge b_0$, consider the function $\psi = 1/\varphi = u/u'$. A calculation gives

$$\psi'(x) = 1 - \frac{u(x)u''(x)}{u'(x)^2} = 1 + p(x)\psi(x)^2 \ge 1 - \tfrac{1}{\varepsilon^2}p(x)^-$$

for $b_0 \le x < \infty$. Integration gives

$$\psi(x) - \psi(b_0) \ge \int_{b_0}^x \left(1 - \tfrac{1}{\varepsilon^2}p(t)^-\right)dt \ge x - b_0 - \tfrac{1}{\varepsilon^2}\int_{b_0}^x p(t)^-\,dt\,,$$

which implies that $\psi(x) > 0$ for sufficiently large x, a contradiction. Therefore, $u(x)$ must vanish somewhere in the interval $(0,\infty)$. $\square$

Applying Theorem 3 to the equation (6) with $\ell = \infty$ and appealing to the relative convexity lemma, we obtain the following result, where the integrals are written in terms of the variable $x = G(y)$.

Theorem 4. *Let p and q be continuous functions on an interval $[a,b)$, where $b \le \infty$. Let u be a solution of $u'' + pu = 0$ such that $u(x) > 0$ on $[a,b)$ and $\int_a^b 1/u(x)^2\,dx = \infty$, and let v be a solution of $v'' + qv = 0$ with $v(a) = u(a)$ and $v'(a) = u'(a)$. If*

$$\int_a^b \big(q(x) - p(x)\big)^- u(x)^2\,dx < \int_a^b \big(q(x) - p(x)\big)^+ u(x)^2\,dx \le \infty\,,$$

then v vanishes at some point in (a,b).

For example, let $p(x) = 1/(1-x^2)^2$ and $u(x) = \sqrt{1-x^2}$ for $x \in [0,1)$. By Theorem 4, the solution of

$$v''(x) + \left(\frac{1}{(1-x^2)^2} + r(x)\right)v(x) = 0\,, \qquad v(0) = 1, \quad v'(0) = 0\,,$$

will vanish somewhere in $(0,1)$ if $\int_0^1 (1-x^2)\,r(x)^-\,dx < \int_0^1 (1-x^2)\,r(x)^+\,dx\,.$

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